HEAT TRANSFER IN LAMINAR FLOW. I. POWER-LAW FLUID IN ANNULAR DUCT

K. WICHTERLE, O. WEIN and J. ULBRECHT*

Institute of Chemical Process Fundamentals, Czechoslovak Academy of Sciences, Prague-Suchdol

Received October 23rd, 1970

The solution of the Grätz–Nusselt problem for flow of a non-Newtonian power-law liquid in annular duct is presented, together with some numerical results enabling description of heat transfer in exchangers with dimensionless length exceeding the value 0·01.

The original Grätz–Nusselt problem is related with the formulation and solution of the mathematical model describing heat transfer into liquid in laminar flow through a pipe with a step temperature[1] change taking place on its wall. This term is also used in general for other problems related to heat transfer in liquids flowing in ducts of various cross-sections. These problems are usually limited to the laminar flow and to heat transfer by axial convection and by radial conduction from the walls. Solution of the Grätz–Nusselt problems serves to several practical purposes: to estimation of the effect of forced convection on temperature regime of heat transfer units; to calculation of heat-exchangers in which the conditions approach the assumptions included in the mathematical model of the process; to measurements of thermal conductivity under the same conditions as in the previous case.

We concentrate our attention here to ducts of annular cross-section on whose wall of radius R a steep temperature change takes place while the other wall of radius $\varkappa R$ is insulated and the velocity profile corresponds to the power-law flow model[2]. Some limiting cases of these problems have already been solved in literature.

For the Grätz–Nusselt problem in Newtonian flow through an annular duct an extensive numerical material has been tabelated by Lundberg and coworkers[3]. Let us name at least the study of Brown[4] presenting the most accurate numerical results from a number of papers dealing with the classical Grätz–Nusselt problem for Newtonian pipe-flow. In the case of plug flow the solution leads to a series of Bessel functions[5]; eventually for the flat duct to a series of trigonometrical functions. Heat transfer in power-law liquids flowing through a tube has been solved by Lyche and Bird[6] and Dente[7] by the classical method of separation of variables. We have completed these results, made them more accurate and we presented them in a form unified for all cases of the considered general annular duct, the cases of a tube and a flat channel inclusive.

* Present address: University of Salford, Salford, England.

THEORETICAL

Equation of Heat Transfer

All the mentioned problems, let it be the heat transfer in a pipe, in an annular or a flat duct, are commonly described mathematically by the differential equation in the dimensionless form

$$w(y)\frac{\partial t}{\partial z} = \frac{\partial^2 t}{\partial y^2} - \frac{1-\varkappa}{1-y(1-\varkappa)}\frac{\partial t}{\partial y}, \qquad (1)$$

with boundary conditions

$$t = 1 \quad \text{for} \quad z \leqq 0\,, \quad t = 0 \quad \text{for} \quad y = 0 \text{ and } z > 0\,, \qquad (2, 3)$$

$$\partial t/\partial y = 0 \quad \text{for} \quad y = 1\,. \qquad (4)$$

The geometrical simplex $\varkappa$, distinguishing individual geometrical arrangements, equals to zero for the pipe, to 1 for the flat duct, to $0 < \varkappa < 1$ for the annular duct where the heat transfer takes place on the outside wall, and to $\varkappa > 1$ for the annular duct where heat transfer takes place only on the inside wall. Dimensionless variables for the pipe and the annular duct are defined by relations

$$z = xk/[\varrho_L c_p U R^2 (1-\varkappa)^2]\,, \quad y = (R-r)/[R(1-\varkappa)] \qquad (5a, 6a)$$

and for the flat duct by relations

$$z = xK/\varrho_L c_p U H^2\,, \quad y = h/H\,. \qquad (5b, 6b)$$

Dimensionless velocity is defined for all cases identically as

$$w(y) = v/U\,. \qquad (7)$$

Dimensionless temperature is then defined relative to the initial temperature of the liquid T_0 and the constant temperature of the heat transfer wall T_w as

$$t = (T - T_w)/(T_0 - T_w)\,. \qquad (8)$$

The significance of some of the quantities in individual geometries under consideration is given in Fig. 1.

Solution of Eq. (*1*) can be written as a series

$$t(y, z) = \sum_{i=1}^{\infty} c_i Y_i(y) \exp\left[-b_i^2 z\right], \qquad (9)$$

where $Y_i(y)$ are eigenfunction and b_i^2 are corresponding eigenvalues obtained by solution of the following system of the Sturm–Liouville type:

$$\frac{d^2 Y_i}{d y_2} + \frac{1 - \varkappa}{1 - y(1 - \varkappa)} \frac{dY_i}{dy} + b_i^2 w(y)\, Y_i = 0\,, \tag{10}$$

$$Y_i = 0 \quad \text{for} \quad y = 0\,, \tag{11}$$

$$Y_i = 0 \text{ and } dY_i/dy = 0 \quad \text{for} \quad y = 1\,, \tag{12}$$

where $w(y)$ is the standardized laminar velocity profile.

The values of constants c_i of the series (9) are given by relation

$$c_i = \frac{\int_0^1 Y_i(y)\,[1 - y(1 - \varkappa)]\, w(y)\, dy}{\int_0^1 [Y_i(y)]^2\,[1 - y(1 - \varkappa)]\, w(y)\, dy}\,, \tag{13}$$

resulting from conditions of orthogonality of series $Y_i(y)$.

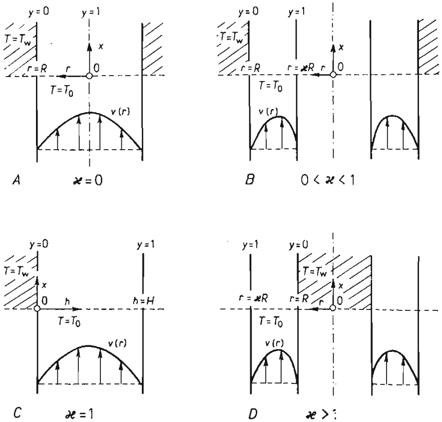

FIG. 1

Example of Heat Transfer Described by Eq. (1) with Boundary Conditions (2)—(4)

A Tube, *B*, *D* annular duct with temperature step change on the outside or inside wall, *C* asymmetrically heated (cooled) flat duct.

The overall temperature field interests an engineer only exceptionally. Usually, the sufficient result of solution of the problem is some of integral heat transfer quantities for instance the dimensionless mixed mean temperature. It is defined as $t_M = (T_M - T_W)/(T_0 - T_w)$ and if the temperature field is known it can be calculated by integration according to

$$t_M(z) = \frac{2}{1 + \varkappa} \int_0^1 t(y, z)\, w(y)\, [1 - y(1 - \varkappa)]\, dy\,. \tag{14}$$

Temperature T_M is the temperature which the liquid flowing out from a given cross-section of the exchanger (z = const.) should have after adiabatic homogenisation. It is obvious that the knowledge of this quantity is sufficient for temperature balance of the exchanger or its parts.

On combining Eqs (*9*) and (*14*) we get the series

$$t_M(z) = \sum_{i=1}^{\infty} t_{Mi} \exp[-b_i^2 z]\,, \tag{15}$$

where coefficients t_{Mi} are

$$t_{Mi} = \frac{2c_i}{1 + \varkappa} \int_0^1 Y_i(y)\, [1 - y(1 - \varkappa)]\, w(y)\, dy\,. \tag{16}$$

For determination of constants t_{Mi}, b_i^2 it is however necessary to determine at first the functions $Y_i(y)$, *i.e.* to solve the above mentioned Sturm–Liouville system (*10*) for the given boundary conditions (*11*) and (*12*).

Calculation Algorithm

The velocity profile of the power-law liquids in annular flow has been obtained by integration[2]

$$w = \frac{(1 - \varkappa^2) \int_0^p |\varrho - \lambda^2/\varrho|^{n-1} (\varrho - \lambda^2/\varrho)\, d\varrho}{2 \int_0^1 \xi \int_0^\xi |\varrho - \lambda^2/\varrho|^{n-1} (\varrho - \lambda^2/\varrho)\, d\varrho\, d\xi}\,, \tag{17}$$

where

$$\varrho = 1 - y(1 - \varkappa)\,. \tag{17}$$

The value λ was calculated by integration of the differential equation[8]

$$\frac{d\lambda}{d\varkappa} = \lambda \Big/ \left\{ \varkappa + \left[\frac{\varkappa - (1 - \lambda)^2}{\lambda^2 - \varkappa^2} \right]^{1/n} \right\}, \tag{19}$$

with initial conditions

$$\lambda = 1 \quad \text{and} \quad d\lambda/d\varkappa = 1/2 \quad \text{for} \quad \varkappa = 1\,. \tag{20}$$

For calculation of eigenvalues and eigenfunctions an iteration procedure based on functional properties of the Sturm–Liouville systems described by Berry and de Prima[9] was used. If we choose an arbitrary $[b_i^2]_j$, we can find the function $[Y_i(0)]_j$ and its derivation $[Y_i'(0)]_j$ in the point $y = 0$ by integration of Eq. (*10*) with initial conditions (*12*). If there is a disagreement with the condition (*11*) the procedure can be repeated with the characteristic value

$$[b_i^2]_{j+1} = [b_i^2]_j + \frac{[Y_i(0)]_j\,[Y_i'(0)]_j}{\int_0^1 [Y_i(y)]_j\,[1 - y(1 - \varkappa)]\,w(y)\,dy}\,, \tag{21}$$

until a sufficient accuracy is reached. The sequence $\{[b_i^2]_j\}$, $j = 0, 1, 2 \ldots$ converges to one of the characteristic values b_i^2, while i is the number of local extremes of the function $Y_i(y)$ in the interval $0 \leqq y \leqq 1$. Since it is desirable to make each calculation only once, the choice of the first approximation is very important. When there have not yet been enough data enabling inter- or extrapolation, the estimation based on the WKBJ method[10]

$$[b_i]_0 = \frac{\pi(2i - 1)}{2\int_0^1 [w(y)]^{1/2}\,dy}\,, \tag{22}$$

which leads to very accurate estimates of the higher and to suitable estimates of the lower characteristic values proved to be very useful.

Calculations were performed on the digital computer NE 503. For integration of differential equations, the standard Runge–Kutta–Merson subroutine was used, for the quadratures the Romberg method was used with division of the interval into 32 parts. The velocity profile calculated according to relation (*17*) was substituted by 32 interpolation third-order parabolas. The algorithm used ensures accuracy of five decimals in the result.

RESULTS

We have tabulated[11] the velocity profiles, three eigenfunctions and eigenvalues, and further the coefficients of series (*9*) and (*15*) for 36 combinations of parameters $\varkappa$ and n. The constants necessary for substitution into the first three terms of the series (*15*) for calculation of the dimensionless mean mixing temperature are given in Table I.

TABLE I

Constants of First Three Terms of Series (*15*)

n	b_1^2	b_2^2	b_3^2	t_{M1}	t_{M2}	t_{M3}
			$\varkappa = 0$			
0	5·783	30·471	74·887	0·692	0·131	0·053
0·1	4·940	26·696	66·906	0·768	0·122	0·042
0·25	4·354	24·817	62·900	0·798	0·108	0·036
0·5	3·949	23·456	59·698	0·812	0·101	0·034
0·75	3·763	22·745	58·009	0·817	0·099	0·033
1	3·657	22·305	56·961	0·819	0·098	0·033
			$\varkappa = 0·1$			
0	4·854	26·555	67·011	0·697	0·131	0·053
0·1	4·302	24·044	61·867	0·764	0·127	0·043
0·25	3·963	23·343	61·391	0·793	0·112	0·037
0·5	3·785	23·707	62·832	0·818	0·101	0·033
0·75	3·743	24·367	64·408	0·828	0·095	0·031
1	3·740	24·971	65·687	0·835	0·091	0·029
			$\varkappa = 0·25$			
0	4·004	24·345	63·989	0·718	0·127	0·049
0·1	3·693	22·974	61·365	0·776	0·122	0·041
0·25	3·505	22·885	62·286	0·809	0·108	0·034
0·5	3·410	23·552	64·194	0·832	0·095	0·029
0·75	3·388	24·189	65·515	0·843	0·089	0·028
1	3·386	24·686	66·446	0·849	0·084	0·027
			$\varkappa = 0·5$			
0	3·218	23·060	62·552	0·755	0·114	0·042
0·1	3·092	22·511	61·709	0·801	0·110	0·036
0·25	3·012	22·699	63·093	0·831	0·097	0·029
0·5	2·969	23·343	64·881	0·853	0·085	0·025
0·75	2·957	23·857	65·935	0·863	0·079	0·024
1	2·953	24·235	66·625	0·869	0·075	0·023
			$\varkappa = 0·75$			
0	2·767	22·517	61·997	0·786	0·101	0·037
0·1	2·722	22·347	61·937	0·823	0·098	0·032
0·25	2·688	22·588	63·288	0·049	0·087	0·026
0·5	2·666	23·140	64·909	0·869	0·077	0·022
0·75	2·657	23·569	65·845	0·878	0·071	0·021
1	2·652	23·883	66·451	0·884	0·067	0·020
			$\varkappa = 1$			
0	2·467	22·207	61·685	0·811	0·090	0·032
0·1	2·464	22·237	62·014	0·840	0·088	0·029
0·25	2·455	22·474	63·222	0·863	0·079	0·024
0·5	2·443	22·945	64·703	0·881	0·069	0·020
0·75	2·436	23·315	65·586	0·890	0·064	0·019
1	2·430	23·591	66·162	0·896	0·060	0·018

The typical shape of eigenfunctions $Y_i(y)$ is plotted in Fig. 2. The temperature field $t(y, z)$ for the same case is in Fig. 3.

Series (*9*) and (*15*) converge the faster the greater is the value z. However, already at $z = 0{\cdot}01$ the series with only the first three terms ensures the determination of dimensionless temperature with an accuracy $\pm 0{\cdot}05$. The character of deviations of the temperature field calculated with the use of a final number of terms of the series (*9*) from the exact solution is obvious from the example given in Fig. 3.

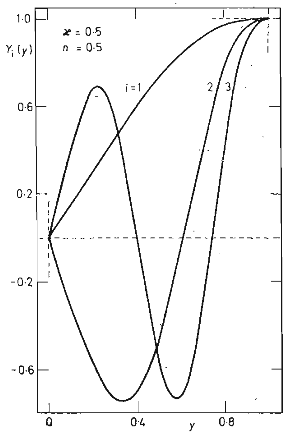

FIG. 2

Plot of First Three Characteristic Functions $Y_i(y)$ for the Case $\varkappa = 0{\cdot}5$; $n = 0{\cdot}5$
$c_1 = 1{\cdot}321$, $c_2 = -0{\cdot}495$, $c_3 = 0{\cdot}295$.

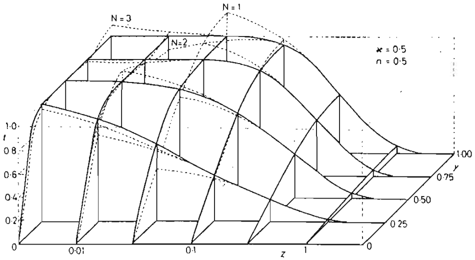

FIG. 3

Temperature Field for the Case $\varkappa = 0{\cdot}5$, $n = 0{\cdot}5$

Solid line denotes exact solution, dashed line the areas calculated according to relation $t_N = \sum_{i=1}^{N} c_i Y_i(y) \exp[-b_i^2 z]$ for N = 1, 2, 3.

As Reynolds and coworkers[12] have shown for the case of Newtonian liquid flowing through the annular duct, it is possible to calculate, with the knowledge of solution of the Grätz–Nusselt problem for simple boundary conditions (the so-called fundamental solution), the temperature field even for rather complex boundary conditions by mere quadratures without solving again the differential equation. Our solution also belongs to fundamental solutions and enables even at boundary conditions

$$t = t_0(x)\,, \quad \text{for} \quad z = 0\,,$$

$$t = t_w(z)\,, \quad \text{for} \quad y = 0 \text{ and } z = 0\,,$$

$$\partial t/\partial y = f(z)\,, \quad \text{for} \quad y = 1\,,$$

to determine the temperature field and the mixed mean temperatures.

LIST OF SYMBOLS

c_p	specific heat (cal g^{-1} deg^{-1})
h	distance from heat transfer wall (cm)
H	distance of walls of flat duct (cm)
k	heat conductivity of liquid (cal cm^{-1} s^{-1} deg^{-1})
r	distance from axis of symmetry (cm)
R	radius of wall at which a step temperature change takes place (cm)
T	temperature (°C)
T_M	mixed mean temperature (°C)
T_0	inlet temperature of liquid (°C)
T_W	wall temperature (°C)
U	mean velocity (cm s^{-1})
v	point velocity (cm s^{-1})
x	axial distance from the location of step temperature change
ϱ_L	liquid density (g cm^{-3})

Dimensionless quantities

b_i^2	eigenvalues	w	velocity, Eq. (*7*)
c_i	expansion coefficient, Eq. (*9*)	y	transverse coordinate, Eq. (*6*)
f	arbitrary function	z	axial coordinate, Eq. (*5*)
n	flow index	Y_i	eigenfunctions
t	temperature, Eq. (*8*)	$\varkappa$	geometric simplex of annulus (ratio of wall radii)
t_M	mixed mean temperature, Eq. (*14*)	λ	characteristic value of velocity profile
t_{Mi}	expansion coefficient, Eq. (*15*)	ϱ	dimensionless radius, Eq. (*18*)
t_0, t_W	arbitrary function	ξ	integration variable

REFERENCES

1. Jakob M.: *Heat Transfer*, Vol. I. Wiley, New York 1957.
2. Fredrickson A. G., Bird R. B.: Ind. Eng. Chem. *50*, 347 (1958).
3. Lundberg R. E., McCuen P. A., Reynolds W. C.: Int. J. Heat Mass Transfer *6*, 495 (1963).
4. Brown G. M.: A.I.CH.E. J. *6*, 170 (1960).
5. Schmidt H. J.: Int. J. Heat Mass Transfer *6*, 719 (1963).
6. Lyche B. C., Bird R. B.: Chem. Eng. Sci. *6*, 49 (1960).
7. Dente N.: Rend. Sci. Fis. Mat. Nat. *29*, 52 (1960).
8. Wein O., Nebřenský J., Wichterle K.: Rheol. Acta *9*, 278 (1970).
9. Berry V. J., de Prima C. R.: J. Appl. Phys. *23*, 195 (1952).
10. Ziegenhagen A.: Int. J. Heat Mass Transfer *8*, 499 (1965).
11. Wichterle K.: *Thesis*. Czechoslovak Academy of Sciences, Prague 1968.
12. Reynolds W. C., McCuen P. A., Lundberg R. E.: Int. J. Heat Mass Transfer *6*, 483 (1963).

Translated by M. Rylek.